

Derivations and automorphisms of a Lie algebra of Block type¹

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Abstract. Let \mathcal{B} be the Lie algebra of Block type with basis $\{L_{\alpha,i} \mid \alpha, i \in \mathbb{Z}, i \geq 0\}$ and relations $[L_{\alpha,i}, L_{\beta,j}] = ((\alpha - 1)(j + 1) - (\beta - 1)(i + 1)) L_{\alpha+\beta, i+j}$. In the present paper, the derivation algebra and the automorphism group of \mathcal{B} are explicitly described. In particular, it is shown that the outer derivation space is 1-dimensional and the inner automorphism group of \mathcal{B} is trivial.

Key words: Lie algebras of Block type; derivation; automorphism.

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1. Introduction

Since a class of infinite dimensional simple Lie algebras was introduced by Block [B], generalizations of Lie algebras of this type (usually referred to as *Lie algebras of Block type*) have been studied by many authors (see for example, [DZ, OZ, S1, S2, SZho, WT1, WT2, X1, X2, Z, ZM]). Thanks to their relation to the Virasoro algebra, these algebras have attracted more and more attention in the literature. See for example a survey paper [S4] on quasifinite representations.

The author in [S2] studied the quasifinite representation of a family of Lie algebras of this type $\mathcal{B}(s, G)$ with basis $\{x^{\alpha,i} \mid \alpha \in G, i \in \mathbb{Z}, i \geq 0\}$ over an algebraically closed field \mathbb{F} of characteristic zero and relations

$$[x^{\alpha,i}, x^{\beta,j}] = s(\beta - \alpha)x^{\alpha+\beta, i+j} + ((\alpha - 1 + s)j - (\beta - 1 + s)i)x^{\alpha+\beta, i+j-1}, \quad (1.1)$$

where G is a nonzero additive subgroup of \mathbb{F} and $s = 0, 1$.

In [S2], it is pointed out that in case $s = 0$, the Lie algebra $\mathcal{B}(0, G)$ with $2 \in G$ has a nontrivial central extension induced by the following 2-cocycle

$$\phi(x^{\alpha,i}, x^{\beta,j}) = (\alpha - 1)\delta_{\alpha+\beta, 2}\delta_{i,0}\delta_{j,0}c, \quad (1.2)$$

where c is a central element. By taking $L_{\alpha,i} = x^{\alpha, i+1}$ in $\mathcal{B}(0, G)$, we see that the Lie brackets in (1.1) take the following form

$$[L_{\alpha,i}, L_{\beta,j}] = ((\alpha - 1)(j + 1) - (\beta - 1)(i + 1)) L_{\alpha+\beta, i+j} \quad \text{for } \alpha, \beta \in G, i, j \geq -1. \quad (1.3)$$

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In this paper, we focus on the Lie subalgebra \mathcal{B} of $\mathcal{B}(0, \mathbb{Z})$, with basis $\{L_{\alpha,i} \mid \alpha \in \mathbb{Z}, i \geq 0\}$ and the above relations. The motivation to study this special Block type Lie algebra is mainly based on a fact that the central extension, denoted $\widehat{\mathcal{B}}$, of \mathcal{B} , which is completely different from $\mathcal{B}(0, \mathbb{Z})$ (see (1.2)), is given by

$$[L_{\alpha,i}, L_{\beta,j}] = ((\alpha - 1)(j + 1) - (\beta - 1)(i + 1)) L_{\alpha+\beta, i+j} + \delta_{\alpha+\beta, 0} \delta_{i,0} \delta_{j,0} \frac{\alpha^3 - \alpha}{6} c,$$

for $\alpha, \beta \in \mathbb{Z}$, $i, j \geq 0$, and which contains a subalgebra with basis $\{L_{\alpha,0}, c \mid \alpha \in \mathbb{Z}\}$ isomorphic to the well-known Virasoro algebra, whereas no central extensions of $\mathcal{B}(0, \mathbb{Z})$ can contain such a subalgebra. Because of this, one may expect that the representation theory of $\widehat{\mathcal{B}}$ will be richer and more interesting than that of $\mathcal{B}(0, \mathbb{Z})$ or its central extension.

We realize that the Lie algebra \mathcal{B} is in fact isomorphic to the Lie algebra defined in [WT1, WT2] (by regarding $L_{\alpha-i,i}$ defined here as $-L_{\alpha,i}$ defined there). Thus central extensions, modules of the intermediate series and quasifinite irreducible highest weight modules of \mathcal{B} have been considered in [WT1, WT2]. However, the problem of classification of quasifinite irreducible \mathcal{B} -modules (which is definitely an important problem in the representation theory) remains open. It is well understood that the representation theory of a Lie algebra often depends on its structure theory. The aim of the present paper is to further study the structure theory of \mathcal{B} in order to obtain sufficient information to give a classification of quasifinite irreducible \mathcal{B} -modules in the future. In this paper, we first characterize the structure of the derivation algebra of \mathcal{B} and prove that the outer derivation space or the first cohomology group of \mathcal{B} with coefficients in its adjoint module is 1-dimensional (see Theorem 2.1). Then we determine the automorphism group of \mathcal{B} and show that \mathcal{B} has no nontrivial inner automorphisms (see Theorem 3.6). Finally, we would like to point out that although \mathcal{B} is \mathbb{Z} -graded with respect to eigenvalues of $\text{ad}_{L_{0,0}}$, it is not finitely-generated \mathbb{Z} -graded, some classical methods (e.g., that in [F]) cannot be applied in our case here.

2. Derivations of \mathcal{B}

Recall that a *derivation* d of the Lie algebra \mathcal{B} is a linear transformation on \mathcal{B} such that

$$d([x, y]) = [d(x), y] + [x, d(y)] \quad \text{for } x, y \in \mathcal{B}.$$

Denote by $\text{Der } \mathcal{B}$ the space of the derivations of \mathcal{B} and $\text{ad } \mathcal{B}$ the space of the *inner derivations* of \mathcal{B} . It is well known that $\text{Der } \mathcal{B}$ forms a Lie algebra with respect to the commutators of linear transformation of \mathcal{B} and $\text{ad } \mathcal{B}$ is an ideal of $\text{Der } \mathcal{B}$. Elements in $\text{Der } \mathcal{B} \setminus \text{ad } \mathcal{B}$ are called *outer derivations*. The *outer derivation space* of \mathcal{B} or the *first cohomology group of \mathcal{B} with coefficients in its adjoint module* is defined by

$$H^1(\mathcal{B}) = \text{Der } \mathcal{B} / \text{ad } \mathcal{B}.$$

Note that $\mathcal{B} = \oplus_{\alpha \in \mathbb{Z}} \mathcal{B}_\alpha$ is a \mathbb{Z} -graded Lie algebra with $\mathcal{B}_\alpha = \text{span}\{L_{\alpha,i} \mid i \in \mathbb{Z}_+\}$. For $\alpha \in \mathbb{Z}$, $i \in \mathbb{Z}_+$, we give the following notations

$$\begin{aligned} \mathcal{B}_\alpha^{[i]} &= \text{span}\{L_{\alpha,j} \mid j \leq i\}, \quad \mathcal{B}_\alpha^{(i)} = \text{span}\{L_{\alpha,j} \mid j < i\}, \\ (\text{Der } \mathcal{B})_\alpha &= \{d \in \text{Der } \mathcal{B} \mid d(\mathcal{B}_\beta) \subset \mathcal{B}_{\alpha+\beta} \text{ for } \beta \geq 0\}. \end{aligned}$$

In particular, $\text{Der } \mathcal{B} = \oplus_{\alpha \in \mathbb{Z}} (\text{Der } \mathcal{B})_\alpha$ is \mathbb{Z} -graded. Obviously, we have a homogeneous derivation of \mathcal{B} defined by

$$d_0 : L_{\beta,j} \mapsto \beta L_{\beta,j} \quad \text{for } \beta \in \mathbb{Z}, j \in \mathbb{Z}_+, \quad (2.1)$$

which can be easily verified to be an outer derivation.

Theorem 2.1 *The \mathbb{Z} -graded derivation algebra $\text{Der } \mathcal{B} = \oplus_{\alpha \in \mathbb{Z}} (\text{Der } \mathcal{B})_\alpha$ has the following decomposition:*

$$\text{Der } \mathcal{B} = \text{ad } \mathcal{B} \oplus \text{D}, \quad \text{where } \text{D} = \text{span}\{d_0\}.$$

In particular, the first cohomology group of \mathcal{B} is 1-dimensional, namely, $\dim H^1(\mathcal{B}) = 1$.

Proof. Let $d \in \text{Der } \mathcal{B}$. The proof of the theorem is equivalent to proving that d is spanned by $\text{ad}_u \in \text{ad } \mathcal{B}$ for some $u \in \mathcal{B}$ and $d_0 \in \text{D}$. This will be done by the following two lemmas (Lemma 2.2 and 2.3). \square

For a fixed integer $\alpha \in \mathbb{Z}$, consider a nonzero derivation $d \in (\text{Der } \mathcal{B})_\alpha$ such that

$$d(\mathcal{B}^{[j]}) \subset \mathcal{B}^{[i+j]} \quad \text{for any } j \in \mathbb{Z}_+, \quad (2.2)$$

where $i \in \mathbb{Z}$ is a fixed integer. Using the similar technique as in [SZho], we assume that the integer i is the minimal one satisfying (2.2). Then we can write

$$d(L_{\beta,j}) \equiv e_{\beta,j} L_{\alpha+\beta,i+j} \pmod{\mathcal{B}^{(i+j)}}, \quad (2.3)$$

where $e_{\beta,j} \in \mathbb{F}$ and we adopt the convention that if a notation is not defined but technically appears in an expression, we always treat it as zero; for example, $e_{1,0} = 0$ if $i < 0$ in (2.2).

Applying d to $[L_{\beta,j}, L_{\gamma,k}] = ((\beta - 1)(k + 1) - (\gamma - 1)(j + 1)) L_{\beta+\gamma,j+k}$, we have

$$\begin{aligned} & ((\alpha + \beta - 1)(k + 1) - (\gamma - 1)(i + j + 1)) e_{\beta,j} \\ & + ((\beta - 1)(i + k + 1) - (\alpha + \gamma - 1)(j + 1)) e_{\gamma,k} \\ & = ((\beta - 1)(k + 1) - (\gamma - 1)(j + 1)) e_{\beta+\gamma,j+k}. \end{aligned} \quad (2.4)$$

Claim 1. We can assume that $i \in \mathbb{Z}_+$ in (2.2).

Otherwise, if $i < 0$, then $e_{1,0} = 0$ as stated above. Taking $\gamma = 1, k = 0$ in (2.4), we have

$$(\alpha + \beta - 1)e_{\beta,j} = (\beta - 1)e_{\beta+1,j},$$

which implies that $e_{\beta,j}$ does not depend on j for any β . Letting $j = 0$ in (2.3), we obtain that $i \geq 0$ by the assumption on the minimality of i , a contradiction.

Lemma 2.2 *If $\alpha + i \neq 0$ or $\alpha + i = 0$ with $i \neq 0$, then d in (2.2) is an inner derivation.*

Proof. For the case $\alpha + i \neq 0$, taking $\gamma = k = 0$ in (2.4), we have

$$(\alpha + i)e_{\beta,j} = ((\alpha - 1)(j + 1) - (\beta - 1)(i + 1))e_{0,0}. \quad (2.5)$$

Set $u_1 = (\alpha + i)^{-1}e_{0,0}L_{\alpha,i} \in \mathcal{B}$ and let $d' = d - \text{ad}_{u_1}$. From (1.3) and (2.5) we see that $d'(L_{\beta,j}) \in \mathcal{B}^{(i+j)}$ for $\beta \in \mathbb{Z}, j \in \mathbb{Z}_+$. Now by induction on i , one can derive that d' is an inner derivation, and then $d = d' + \text{ad}_{u_1}$ is also an inner derivation.

For the other case $\alpha + i = 0$ with $i \neq 0$, we see immediately that $e_{0,0} = 0$ by (2.5). Applying d to $[L_{\beta-1,j}, L_{1,0}] = (\beta - 2)L_{\beta,j}$ and $[L_{\beta,j}, L_{-1,0}] = (\beta + 2j + 1)L_{\beta-1,j}$ respectively, we obtain

$$(\beta - i - 2)e_{\beta-1,j} + ((\beta - 2)(i + 1) + i(j + 1))e_{1,0} = (\beta - 2)e_{\beta,j}, \quad (2.6)$$

$$(\beta + i + 2j + 1)e_{\beta,j} + ((\beta - 1)(i + 1) + (i + 2)(j + 1))e_{-1,0} = (\beta + 2j + 1)e_{\beta-1,j}. \quad (2.7)$$

In particular, taking $\beta = j = 0$ in (2.6), we see that

$$e_{-1,0} + e_{1,0} = 0. \quad (2.8)$$

Multiplying (2.6) by $\beta + 2j + 1$, (2.7) by $\beta - i - 2$, and then adding both results together, we obtain $i(i + 2j + 3)(e_{\beta,j} - (\beta + j)e_{1,0}) = 0$ by (2.8), which implies for $i \neq 0$ that

$$e_{\beta,j} = (\beta + j)e_{1,0} \quad \text{for } \beta \in \mathbb{Z}, j \in \mathbb{Z}_+. \quad (2.9)$$

Set $u_2 = -\frac{1}{i+1}e_{1,0}L_{-i,i} \in \mathcal{B}$ and let $d'' = d - \text{ad}_{u_2}$. By (1.3) and (2.9), we obtain that $d''(L_{\beta,j}) \in \mathcal{B}^{(i+j)}$ for $\beta \in \mathbb{Z}, j \in \mathbb{Z}_+$. As in the first case, by induction on i , we see that d'' is an inner derivation, thus d is also an inner derivation. \square

Lemma 2.3 *If $\alpha = i = 0$, then d in (2.2) can be written as $d = \text{ad}_u + \lambda d_0$ for some $u \in \mathcal{B}$ and $\lambda \in \mathbb{F}$.*

Proof. Now the equations (2.6) and (2.7) can be simplified as

$$(\beta - 2)(e_{\beta-1,j} + e_{1,0} - e_{\beta,j}) = 0, \quad (2.6')$$

$$(\beta + 2j + 1)(e_{\beta-1,j} - e_{-1,0} - e_{\beta,j}) = 0. \quad (2.7')$$

We claim that

$$e_{\beta,j} = \beta e_{1,0} + e_{0,j} \quad \text{for } \beta \in \mathbb{Z}, j \in \mathbb{Z}_+. \quad (2.10)$$

In fact, if $\beta \neq 2$, then $e_{\beta,j} = e_{1,0} + e_{\beta-1,j}$ by (2.6'). By induction on β , one can easily obtain that

$$e_{\beta,j} = \begin{cases} \beta e_{1,0} + e_{0,j} & \text{if } \beta \leq 1, \\ (\beta - 2)e_{1,0} + e_{2,j} & \text{if } \beta \geq 3. \end{cases} \quad (2.11)$$

If $\beta = 2$, then $e_{2,j} = e_{1,j} - e_{-1,0} = e_{1,j} + e_{1,0} = 2e_{1,0} + e_{0,j}$ by (2.7'), (2.8) and the first case of (2.11) respectively. This, together with (2.11), gives the claim.

On the other hand, the equation (2.4) can be rewritten as $((\beta - 1)(k + 1) - (\gamma - 1)(j + 1))(e_{\beta,j} + e_{\gamma,k} - e_{\beta+\gamma,j+k}) = 0$. Substituting (2.10) in this formula gives

$$((\beta - 1)(k + 1) - (\gamma - 1)(j + 1))(e_{0,j} + e_{0,k} - e_{0,j+k}) = 0.$$

Then $e_{0,j+k} = e_{0,j} + e_{0,k}$ by arbitrariness of β or γ . By induction on j , one can derive that $e_{0,j} = j e_{0,1}$, which, together with (2.10), gives

$$e_{\beta,j} = \beta e_{1,0} + j e_{0,1} \quad \text{for } \beta \in \mathbb{Z}, j \in \mathbb{Z}_+. \quad (2.12)$$

Set

$$\bar{d} = d + \text{ad}_{u_3} - (e_{1,0} - e_{0,1})d_0,$$

where $u_3 = e_{0,1}L_{0,0} \in \mathcal{B}$ and d_0 is defined by (2.1). Applying \bar{d} to the formula $[L_{0,0}, L_{\beta,j}] = -(\beta + j)L_{\beta,j}$, using (2.12), we obtain that $\bar{d}(L_{\beta,j}) \in \mathcal{B}^{(j)}$ for $\beta \in \mathbb{Z}, j \in \mathbb{Z}_+$. By Lemma 2.2, \bar{d} is an inner derivation, and then $d = \text{ad}_u + (e_{1,0} - e_{0,1})d_0$ for some $u \in \mathcal{B}$. This completes the proof. \square

3. Automorphisms of \mathcal{B}

An element $S \in \mathcal{B}$ is called

- (i) *ad-locally finite* if for any given $v \in \mathcal{B}$, the subspace $\text{Span}\{\text{ad}_S^m \cdot v \mid m \in \mathbb{Z}_+\}$ of \mathcal{B} is finite dimensional,
- (ii) *ad-locally nilpotent* if for any given $v \in \mathcal{B}$, there exists some $N > 0$ such that $\text{ad}_S^N \cdot v = 0$.

Denote by $\text{Aut } \mathcal{B}$ the *automorphism group* of \mathcal{B} , and $\text{Int } \mathcal{B}$ the *inner automorphism group* of \mathcal{B} , namely, the subgroup of $\text{Aut } \mathcal{B}$, generated by \exp^{ad_x} for ad-locally nilpotent elements x 's.

In this section, we first prove that \mathcal{B} does not have a nonzero locally nilpotent element, thus the inner automorphism group of \mathcal{B} is trivial. Next we construct three kinds of outer automorphisms of \mathcal{B} , and then completely characterize the structure of the automorphism group of the Lie algebra \mathcal{B} .

Lemma 3.1 *Up to scalars, $L_{0,0}$ is the unique locally finite element of \mathcal{B} . Furthermore, \mathcal{B} does not have a nonzero locally nilpotent element, thus the inner automorphism group of \mathcal{B} is trivial.*

Proof. Take any locally finite element $S = \sum_{(\alpha,i) \in I_S} \lambda_{\alpha,i} L_{\alpha,i}$ of \mathcal{B} , where I_S is a finite subset of $\mathbb{Z} \times \mathbb{Z}_+$. First, suppose that there exists $\lambda_{\alpha,i} \neq 0$ for some $\alpha < 0$. Take the minimal $\alpha_0 < 0$ such that there exists some i with $\lambda_{\alpha_0,i} \neq 0$, and then choose $i = i_0$ to be the maximal one satisfying this condition. By rescaling S , we may suppose

$$S = L_{\alpha_0,i_0} + \sum_{\substack{\alpha > \alpha_0 \text{ or} \\ \alpha = \alpha_0, i < i_0}} \lambda_{\alpha,i} L_{\alpha,i},$$

and in this case we say that S has the *minimal term* L_{α_0,i_0} . Recall that $[L_{\alpha_0,i_0}, L_{\beta,j}] = F_{\beta}^j L_{\alpha_0+\beta,i_0+j}$, where we use the following notation

$$F_{\beta}^j := (\alpha_0 - 1)(j + 1) - (\beta - 1)(i_0 + 1).$$

If $\alpha_0 + i_0 \geq 0$ (or > 0), we can choose big (or small) enough β_0 and suitable j_0 such that $F_{\beta_0}^{j_0} < 0$ (or > 0) and

$$F_{\beta_0+k\alpha_0}^{j_0+ki_0} = F_{\beta_0}^{j_0} - k(\alpha_0 + i_0) < 0 \text{ (or } > 0) \text{ for all } k \in \mathbb{Z}_+,$$

which implies that $\text{ad}_S^k(L_{\beta_0,j_0})$, with minimal terms $L_{\beta_0+k\alpha_0,j_0+ki_0}$, are linear independent for all k , i.e., S is not ad-locally finite. Hence $\lambda_{\alpha,i} = 0$ for all $\alpha < 0$. Similarly, we can also show that $\lambda_{\alpha,i} = 0$ for all $\alpha > 0$.

Now we can rewrite $S = \sum_{i \in I'_S} \lambda_{0,i} L_{0,i}$, where I'_S is a finite subset of \mathbb{Z}_+ . If there exists $\lambda_{0,i} \neq 0$ for some $i > 0$, then similarly take $i_0 > 0$ to be the maximal one and assume that

$$S = L_{0,i_0} + \sum_{i < i_0} \lambda_{0,i} L_{0,i}.$$

Now $[L_{0,i_0}, L_{\beta,j}] = G_{\beta}^j L_{\beta,i_0+j}$, where $G_{\beta}^j = -(j+1) - (\beta-1)(i_0+1)$. One can take big enough β_0 and some j_0 satisfying

$$G_{\beta_0}^{j_0+k i_0} = G_{\beta_0}^{j_0} - k i_0 < 0 \quad \text{for } k \in \mathbb{Z}_+,$$

which also contradicts our assumption. So $\lambda_{0,i} = 0$ for all $i > 0$, and thus $S = \lambda_{0,0} L_{0,0}$ for some $\lambda_{0,0} \in \mathbb{F}$, namely, $L_{0,0}$ is up to scalars the unique locally finite element of \mathcal{B} .

Note that any locally nilpotent element must be locally finite element by definition. Since $\text{ad}_{L_{0,0}}^N L_{\alpha,i} = -N(\alpha+i)L_{\alpha,i} \neq 0$ for any $N > 0$ if $\alpha+i \neq 0$, we know that the locally finite element $L_{0,0}$ is not locally nilpotent. Hence the above statement implies that \mathcal{B} does not have a nonzero locally nilpotent element, and then the inner automorphism group of \mathcal{B} is trivial. \square

Recall that the centerless Virasoro algebra Vir with basis $\{L_{\alpha} | \alpha \in \mathbb{Z}\}$ is defined by the commutation relations: $[L_{\alpha}, L_{\beta}] = (\beta - \alpha)L_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{Z}$. We review a known result about the structure of the automorphism group of Virasoro algebra. It can also be regarded as a corollary of Theorem 2.3 in [SZha].

Proposition 3.2 (1) *For any $\mu \in \mathbb{F}^*$, the following map is an automorphism of Vir .*

$$\chi_{\mu} : \text{Vir} \rightarrow \text{Vir}, \quad L_{\alpha} \mapsto \mu^{\alpha} L_{\alpha} \quad \text{for any } \alpha \in \mathbb{Z}.$$

(2) *For any $s \in \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$, the following map is an automorphism of Vir .*

$$\chi'_s : \text{Vir} \rightarrow \text{Vir}, \quad L_{\alpha} \mapsto s L_{s\alpha} \quad \text{for any } \alpha \in \mathbb{Z}.$$

(3) $\text{Aut}(\text{Vir}) \cong \mathbb{F}^* \rtimes \mathbb{Z}/2\mathbb{Z}$.

Motivated by the above, one can define the following three kinds of maps:

$$\begin{aligned} \varphi_{\mu} : \mathcal{B} &\rightarrow \mathcal{B} & L_{\alpha,i} &\mapsto \mu^{\alpha} L_{\alpha,i}; \\ \varphi'_{\nu} : \mathcal{B} &\rightarrow \mathcal{B} & L_{\alpha,i} &\mapsto \nu^i L_{\alpha,i}; \\ \rho_{\xi} : \mathcal{B} &\rightarrow \mathcal{B} & L_{\alpha,i} &\mapsto \xi L_{\xi(\alpha+i)-i,i}, \end{aligned}$$

where $\mu, \nu \in \mathbb{F}^* = \mathbb{F} \setminus \{0\}$ and $\xi \in \{\pm 1\}$. One can easily check that they are all (outer) automorphisms of \mathcal{B} . Furthermore, we have the following facts.

- (1) $\{\varphi_{\mu} | \mu \in \mathbb{F}^*\} \cong \mathbb{F}^*$ is a subgroup of $\text{Aut} \mathcal{B}$, where $\varphi_{\mu_1} \varphi_{\mu_2} = \varphi_{\mu_1 \mu_2}$ for $\mu_1, \mu_2 \in \mathbb{F}^*$.
- (2) $\{\varphi'_{\nu} | \nu \in \mathbb{F}^*\} \cong \mathbb{F}^*$ is a subgroup of $\text{Aut} \mathcal{B}$, where $\varphi'_{\nu_1} \varphi'_{\nu_2} = \varphi'_{\nu_1 \nu_2}$ for $\nu_1, \nu_2 \in \mathbb{F}^*$.
- (3) $\{\rho_{\xi} | \xi = -1, 1\} \cong \mathbb{Z}/2\mathbb{Z}$ is a subgroup of $\text{Aut} \mathcal{B}$.

Proposition 3.3 *Let $\mathcal{V} = \text{Span}\{L'_\alpha \mid \alpha \in \mathbb{Z}\}$ be a subalgebra of \mathcal{B} , which is isomorphic to the centerless Virasoro algebra, i.e., $[L'_\alpha, L'_\beta] = (\alpha - \beta)L'_{\alpha+\beta}$. Suppose $L'_0 \in \mathbb{F}L_{0,0}$. Then $L'_\alpha \in \mathbb{F}L_{\alpha,0}$ for all $\alpha \in \mathbb{Z}$.*

Proof. By rescaling L'_0 , we can suppose $L'_0 = L_{0,0}$. Let $0 \neq \alpha \in \mathbb{Z}$. Write $L'_\alpha = \sum_{(\beta,j) \in J_\alpha} \mu_{\beta,j} L_{\beta,j}$, where J_α is a finite subset of $\mathbb{Z} \times \mathbb{Z}_+$. Then

$$-\alpha \sum_{(\beta,j) \in J_\alpha} \mu_{\beta,j} L_{\beta,j} = -\alpha L'_\alpha = [L'_0, L'_\alpha] = \left[L_{0,0}, \sum_{(\beta,j) \in J_\alpha} \mu_{\beta,j} L_{\beta,j} \right] = - \sum_{(\beta,j) \in J_\alpha} (\beta + j) \mu_{\beta,j} L_{\beta,j},$$

which implies that $\mu_{\beta,j} = 0$ if $\beta + j \neq \alpha$. Hence we can rewrite

$$L'_\alpha = \sum_{j \in J'_\alpha} \lambda_{\alpha,j} L_{\alpha-j,j}, \text{ where } \lambda_{\alpha,j} = \mu_{\alpha-j,j}, \quad J'_\alpha = \{j \mid (\alpha - j, j) \in J_\alpha\} \subset \mathbb{Z}_+.$$

Then

$$\begin{aligned} 2\alpha L_{0,0} &= [L'_\alpha, L'_{-\alpha}] = \left[\sum_{i \in J'_\alpha} \lambda_{\alpha,i} L_{\alpha-i,i}, \sum_{j \in J'_{-\alpha}} \lambda_{-\alpha,j} L_{-\alpha-j,j} \right] \\ &= \sum_{(i,j) \in J'_\alpha \times J'_{-\alpha}} (i + j + 2) \alpha \lambda_{\alpha,i} \lambda_{-\alpha,j} L_{-(i+j), i+j}. \end{aligned} \quad (3.1)$$

Let $i_0 = \max\{i \mid i \in J'_\alpha, \lambda_{\alpha,i} \neq 0\}$, $j_0 = \max\{j \mid j \in J'_{-\alpha}, \lambda_{-\alpha,j} \neq 0\}$. If $i_0 + j_0 > 0$, then the right-hand side of (3.1) contains the nonzero term $(i_0 + j_0 + 2) \alpha \lambda_{\alpha,i_0} \lambda_{-\alpha,j_0} L_{-(i_0+j_0), i_0+j_0}$, which is not in $\mathbb{F}L_{0,0}$. Thus $i_0 = j_0 = 0$ (since i_0, j_0 are non-negative), in particular $L'_\alpha \in \mathbb{F}L_{\alpha,0}$. \square

Lemma 3.4 *Let $\tau \in \text{Aut } \mathcal{B}$, then $\tau(L_{\alpha,0}) = \xi \mu^\alpha L_{\xi\alpha,0}$ for some $\mu \in \mathbb{F}^*$, and $\xi \in \{\pm 1\}$.*

Proof. Suppose $\tau \in \text{Aut } \mathcal{B}$. Let $L'_\alpha = \tau(L_{\alpha,0})$ for $\alpha \in \mathbb{Z}$. Since $\mathcal{N} = \text{Span}\{L_{\alpha,0} \mid \alpha \in \mathbb{Z}\}$ is the centerless Virasoro algebra, we see that $\mathcal{V} = \text{Span}\{L'_\alpha \mid \alpha \in \mathbb{Z}\}$ is a subalgebra isomorphic to the centerless Virasoro algebra. Furthermore, since $L_{0,0}$ is up to scalars the unique ad-locally finite element in \mathcal{B} , we must have $L'_0 = \tau(L_{0,0}) \in \mathbb{F}L_{0,0}$. So Proposition 3.3 implies $\tau(\mathcal{N}) = \mathcal{V} = \mathcal{N}$. Now the result follows from Proposition 3.2. \square

Lemma 3.5 *Let $\tau \in \text{Aut } \mathcal{B}$, then $\tau(L_{0,i}) = \xi \nu^i L_{(\xi-1)i,i}$ for some $\nu \in \mathbb{F}^*$, and $\xi \in \{\pm 1\}$.*

Proof. Assume

$$\tau(L_{0,i}) = \sum_{(p,q) \in J_i} \nu_{p,q} L_{p,q} \text{ for some } \nu_{p,q} \in \mathbb{F}, \quad (3.2)$$

where J_i is some finite subset of $\mathbb{Z} \times \mathbb{Z}_+$. Applying τ to the equation $[L_{0,0}, L_{0,i}] = -iL_{0,i}$, we get

$$\sum_{(p,q) \in J_i} (i - \xi(p+q)) \nu_{p,q} L_{p,q} = 0,$$

which implies that $\nu_{p,q} = 0$ if $p \neq \xi i - q$. Then (3.2) can be rewritten as

$$\tau(L_{0,i}) = \sum_{q \in J'_i} \lambda_{i,q} L_{\xi i - q, q}, \text{ where } \lambda_{i,q} = \nu_{\xi i - q, q}, \quad J'_i = \{q \mid (\xi i - q, q) \in J_i\}. \quad (3.3)$$

Applying τ to $[L_{-1,0}, [L_{1,0}, L_{0,i}]] = -2(i+1)L_{0,i}$, using Lemma 3.4, we obtain

$$\sum_{q \in J'_i} (q - i + 1)(q + i + 2) \lambda_{i,q} L_{\xi i - q, q} = 2(i+1) \sum_{q \in J'_i} \lambda_{i,q} L_{\xi i - q, q},$$

which then implies that $(q - i)(q + i + 3) \lambda_{i,q} = 0$, and thus $\lambda_{i,q} = 0$ if $q \neq i$. Thus we can rewrite (3.3) as

$$\tau(L_{0,i}) = \xi \nu_i L_{(\xi-1)i, i} \text{ for some } \nu_i \neq 0.$$

Finally, applying τ to the relation $[L_{0,i}, L_{0,1}] = (i-1)L_{0,i+1}$, we obtain $\nu_{i+1} = \nu_i \nu_1$, which implies $\nu_i = \nu^i$, where $\nu = \nu_1$, and the lemma follows. \square

Theorem 3.6 *Let $\tau \in \text{Aut } \mathcal{B}$, then there exist some $\mu, \nu \in \mathbb{F}^*$, $\xi \in \{\pm 1\}$ such that*

$$\tau(L_{\alpha,i}) = \xi \mu^\alpha \nu^i L_{\xi(\alpha+i)-i, i} \text{ for } \alpha, i \in \mathbb{Z}, i \in \mathbb{Z}_+.$$

In particular, $\text{Aut } \mathcal{B} \cong (\mathbb{F}^ \times \mathbb{F}^*) \rtimes \mathbb{Z}/2\mathbb{Z}$.*

Proof. Let $\tau \in \text{Aut } \mathcal{B}$, by Lemma 3.4 and 3.5, we have $\tau(L_{\alpha,0}) = \xi \mu^\alpha L_{\xi\alpha,0}$ and $\tau(L_{0,i}) = \xi \nu^i L_{(\xi-1)i, i}$ for some $\mu, \nu \in \mathbb{F}^*$ and $\xi \in \{\pm 1\}$. Applying τ to the equation $[L_{\alpha,0}, L_{0,i}] = (\alpha(i+1) - i)L_{\alpha,i}$ gives

$$(\alpha(i+1) - i) (\tau(L_{\alpha,i}) - \xi \mu^\alpha \nu^i L_{\xi(\alpha+i)-i, i}) = 0.$$

Thus the result holds if $\alpha(i+1) \neq i$. Assume $\alpha(i+1) = i$, which implies $\alpha = i = 0$ since $i \in \mathbb{Z}_+$. In this case, we have the result by Lemma 3.4. \square

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